Periods and heights of Heegner points

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F = totally real fld. $\mathbb{A} = \mathbb{A}_F$

π = cuspidal rep of $G(\mathbb{A})$.

E/F = quad. extn. $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$

$\omega_\pi \cdot \chi|_{\mathbb{A}_F^\times} = \begin{cases} 1 \\ \eta_{E/F} & \text{quad for } E/F. \end{cases}$

do not know how to do this

no geometry ...

$L(s,\pi,\chi)$ = Rankin - Selberg L-series

$L(s,\pi,\chi) = L(1-s,\pi,\chi)\,\varepsilon(s,\pi,\chi)$

$\varepsilon(\frac{1}{2},\pi,\chi) = \pm 1$ $\quad \varepsilon(\frac{1}{2},\pi,\chi) = \prod \varepsilon_v(\frac{1}{2},\pi_v,\chi_v) = (-1)^{\#\Sigma}$

$\Sigma = \{ v : \text{place of } F \mid \varepsilon_v(\frac{1}{2},\pi_v,\chi_v) \neq \omega_v\eta_v(-1) \}$.

Question: Find a formula for $L(\frac{1}{2},\pi,\chi)$ if $\#\Sigma$ is even

$L'(\frac{1}{2},\pi,\chi)$ if $\#\Sigma$ is odd.

Even case: $\#\Sigma$ = even

B = quat. alg $/F$ w/ ramification at Σ.

$G' = B^\times$ as an alg$/F$

π' = Jacquet-Langlands corresp of π on $G'(\mathbb{A})$

There is an embedding $E \hookrightarrow B$

$E^\times$ as a subgp of G'

$f \in \pi$, $\quad \ell(f,\chi) = \displaystyle\int_{\mathbb{A}_F^\times E^\times \backslash E_{\mathbb{A}}^\times} f(g)\chi(g)\,dg$

Thm (Waldspurger, at least when $\omega_\pi = 1$) $\quad$ Assume π is unitary. up to twist by the central char.

Assume that $f = \otimes_v f_v \in \pi'$ is decomposable

then

$$\frac{|\ell(f,\chi)|^2}{\langle f,f\rangle} = \frac{\zeta_F(2)\,L(\frac{1}{2},\pi,\chi)}{2\,L(1,\pi,\mathrm{ad})} \prod_v \underbrace{c(f_v,\chi_v)}_{\neq 0}$$

(Petersson inner product w.r.t. Tamagawa measure.

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$$c(f_v,\chi_v) = \frac{L(1,\eta_v)\, L(1,\pi_v,\mathrm{ad})}{\zeta(2)\, L(\frac{1}{2},\pi_v,\chi_v)} \int_{F_v^\times\backslash E_v^\times} \frac{\langle \pi_v(t) f_v, f_v\rangle}{\langle f_v, f_v\rangle} \chi_v(t)\, dt$$

$c(f_v,\chi_v) \neq 0$ for all v

$= 1$ for almost all v.

Rmks: For new forms f, same formula was proved by Gross, Shou-wu Zhang.

Odd case: $\# \Sigma =$ odd

Assume $\pi \otimes \chi$ motivic

(1) $\Sigma \supseteq$ all archimedean places $\quad \pi\otimes\chi \leftarrow H^1(X,\mathcal{F})$

Conj

$L'(\frac{1}{2},f,\chi) =$ Height of Heegner cycles in some local system on a Shimura curve

(2) $\Sigma \not\supseteq$ all archimedean places (??)

$F = \mathbb{Q} \qquad \Sigma = \{p\}$

Assume Π is discrete of wt $(2,\ldots,2)$ at archimedean places

χ is a finite character.

$\Sigma \supseteq$ all archimedean places,

$B_{\mathbb{A}}$ = quaternion alg / $\mathbb{A}$. w/ ramification at places in Σ.

For each $\mathcal{U} \subseteq B^\times_{\mathbb{A}_f}$, compact open,

$X_{\mathcal{U}}$ = Shimura curve / F

$X = \varprojlim X_{\mathcal{U}} \curvearrowleft B_{/\mathbb{A}}$.

$\tau : F \hookrightarrow \mathbb{C}$ $B(\tau)/_F$: quaternion alg w/ ramification set $\Sigma \setminus \{\tau\}$

$X_{\mathcal{U},\tau}(\mathbb{C}) = B(\tau)^\times \backslash \mathfrak{h}^\pm \times \hat{B}^\times / \mathcal{U}$ $\widehat{B(\tau)} \simeq \hat{B}$ (First work with $B(\tau)$, then w/ B, indep of τ.)

$E_{\mathbb{A}} \hookrightarrow B_{\mathbb{A}}$ w/ defines a morphism of $C \to X_E$.

$C = \varprojlim C_{\mathcal{U}}$ scheme of zero dim / E

$C(\bar{E})_{\mathcal{U}} = E^\times \backslash \hat{E}^\times / \hat{E}^\times \cap \mathcal{U}$.

$\circlearrowleft$

$\mathrm{Gal}(\bar{E}/_E)$ by class field theory.

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Thm (Yuan - Zhang - Zhang)

Assume that $f = \otimes_v f_v$ decomposable

$$\frac{\langle Y_\chi, T_{f\otimes\bar{f}} Y_\chi \rangle_{NT}}{\langle f, f \rangle} = \frac{\zeta_F(2)\, L'(\frac{1}{2}, \pi, \chi)}{4\, L(1, \pi, ad)} \prod_v C_v(f_v, \chi_v).$$

Rmk: $\omega_\pi = 1$, χ unramified.

- $F = \mathbb{Q}$, $\Sigma = \{\infty\}$ (Heegner condition)

  Thm = GZ under some assumption.

- For general F, $\omega_\pi = 1$ Assumption on ramification

  (S.Z.)

- Ben Howard (more general case)

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Applications:

apply Euler system to get bound on $A_\chi(K)$ ...

...

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Linear forms $\qquad G = GL_2 \times E^\times / \Delta(F^\times)$

$$(\pi, \chi) \rightsquigarrow \Pi \text{ on } G(\mathbb{A})$$

$$T = E^\times / F^\times \longrightarrow G \text{ diagonally}$$

<u>Thm</u> (Tunnel, Waldspurger, Saito)

1) $v \in \Sigma \iff \mathrm{Hom}_{T_v}(\Pi_v, \mathbb{C}) = 0$.

2) $\dim \mathrm{Hom}_{T_v}(\Pi_v, \mathbb{C}) \le 1$.

$G'_{/\mathbb{A}} = B^\times_{/\mathbb{A}} \times E^\times_{/\mathbb{A}} / \Delta(\mathbb{A}^\times)$

$\pi' = $ J-L. on $G'_{/\mathbb{A}}$, (π', χ)

$G'_{/\mathbb{A}}$ is the unique inner form of $G_{/\mathbb{A}}$ s.t. $\mathrm{Hom}(\Pi, \mathbb{C}) \ne 0$.

$$\boxed{\dim \mathrm{Hom}_{T_{\mathbb{A}} \times T_{\mathbb{A}}}(\Pi' \times \tilde{\Pi}', \mathbb{C}) = 1}$$

Construct 3 linear forms α, β, γ.

Waldspurger formula $\qquad \beta = \alpha\, L(\tfrac{1}{2}, \pi)$

GZ $\qquad \gamma = \alpha\, L'(\tfrac{1}{2}, \pi)$.

<u>α</u>: $\qquad f_v \in \Pi'_v, \quad \tilde{f}_v \in \tilde{\Pi}'_v$

$$\int_{T_v^\times} (\Pi'_v(t) f_v, \tilde{f}_v)\, dt = \frac{\xi_v(2)\, L(\tfrac{1}{2}, \Pi_v)}{L(1, \eta_v)\, L(1, \Pi_v, \mathrm{ad})} = A(\Pi_v) \ne 0$$

$$\alpha_v(f, \tilde{f}_v) = A(\Pi_v)^{-1} \int_{\underline{\ \ }^\times} \cdots$$

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$$\alpha \in \mathrm{Hom}_{T_{\mathbb{A}} \times T_{\mathbb{A}}}(\pi' \times \tilde{\pi}', \mathbb{C})$$

β: $(\# \Sigma = \text{even})$

$B_{\mathbb{A}} = B \otimes \mathbb{A}$ $\qquad \pi' \subset \mathcal{A}(G'(F) \backslash G'(\mathbb{A}))$.

$G'_{\mathbb{A}} = G'(\mathbb{A})$ $\qquad f, f' \in \pi'$.

$$\beta(f, f') = \int_{T(F) \backslash T(\mathbb{A})} f(t)\, dt \int_{T(F) \backslash T(\mathbb{A})} \tilde{f}(t)\, dt$$

<u>Waldspurger</u>: $\quad \beta = (*) L(\tfrac{1}{2}, \pi) \alpha$

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Construction of γ ($\#\Sigma$ = odd).

Shimura curve X by $G'_{\mathbb{A}}$.

$T_{\mathbb{A}} \hookrightarrow G'_{E/\mathbb{A}}$

$\rightsquigarrow$. $Y. \longrightarrow X_{E.}$

Shimura variety . for T.

$Y_\xi \in \varprojlim \mathrm{Jac}(X_u)$.

$f, \tilde{f} \in \Pi \otimes \pi'$ $\rightsquigarrow$ $T_{f \circ \tilde{f}}$ a Hecke operator

$\gamma(f, \tilde{f}) := \langle Y_\xi, T_{f\circ\tilde{f}} Y_\xi \rangle_{NT}$. $\in \mathrm{Hom}_{T(\mathbb{A})}(\pi' \otimes \tilde{\pi}', \mathbb{C})$

$\underline{\underline{GZ}}$: $\gamma(f, \tilde{f}) = L'(\tfrac{1}{2}, \pi)\, \alpha(f, \tilde{f})$.

Working with all f's !

=. Gross - Prasad "Linear forms for orthogonal gps"

Ichino & Ikeda ...

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Need to construct kernel functions:

$$\mathcal{S}(B_{\mathbb{A}} \times \mathbb{A}^\times) \otimes \mathcal{A}(E_{\mathbb{A}}^\times / E^\times) \ni \phi$$

$$B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times \times GL_2(\mathbb{A}) \times E_{\mathbb{A}}^\times .$$

fix additive character ψ of $F \backslash \mathbb{A}$

$$\tilde{\Phi}(x,u,t) = \int \phi(xz, uz^{-2}, tz)\, dz$$

$$\tilde{\mathcal{S}} = \{ \tilde{\Phi} \} \ ? \quad \underbrace{G'_{\mathbb{A}} \times G'_{\mathbb{A}}}_{\text{act on } \mathcal{A}(E_{\mathbb{A}}^\times / E^\times)} \times G_{\mathbb{A}}$$

Shimizu Lifting

For any cuspidal π $\quad \mathcal{F} \to \mathbb{C}$

$$\dim \mathrm{Hom}(\tilde{\mathcal{S}}, \tilde{\pi}' \otimes \pi' \otimes \pi) = 1$$

Let $\theta \in \mathrm{Hom}(\tilde{\mathcal{S}}, \tilde{\pi}' \otimes \pi' \otimes \pi)$ be such that $\forall$ $\varphi \in \tilde{\pi}$

$$\langle \varphi, \mathcal{F}\theta(\phi) \rangle = \frac{\zeta(2)}{L(1,\pi,\mathrm{ad})} \int_{N(\mathbb{A}) \backslash GL_2(\mathbb{A})} W_\varphi(g)\, \omega(g) \phi(1,1,1)\, dg \ \ldots$$